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Optimal Control for a Steady State Dead Oil Isotherm Problem*

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Abstract

We study the optimal control of a steady-state dead oil isotherm problem. The problem is described by a system of nonlinear partial differential equations resulting from the traditional modelling of oil engineering within the framework of mechanics of a continuous medium. Existence and regularity results of the optimal control are proved, as well as necessary optimality conditions.

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1 Introduction

We are interested in the optimal control of the steady-state dead oil isotherm problem:

$$\begin{cases} -\Delta\varphi(u) = \operatorname{div}\left(g(u)\nabla p\right) & \text{in } \Omega, \\ -\operatorname{div}\left(d(u)\nabla p\right) = f & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \\ p|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

where Ω is an open bounded domain in $\mathbb{R}^2$ with a sufficiently smooth boundary. Equations (1) serve as a model for an incompressible biphasic flow in a porous medium, with applications to the industry of exploitation of hydrocarbons. The reduced saturation of oil is denoted by u, and p is the global pressure. To understand the optimal control problem that we consider here, some words about the recovery of hydrocarbons are in order. For a more detailed discussion about the physical justification of equations (1) the reader is referred to [4, 16, 17] and references therein. At the time of the first run of a layer, the flow of the crude oil towards the surface is due to the energy stored in the gases under pressure in the natural hydraulic system. To mitigate the consecutive decline of production and the decomposition of the site, water injections are carried out, well before the normal exhaustion of the layer. The water is injected through wells with high pressure, by pumps specially drilled to this end. The pumps allow the displacement of the crude oil towards the wells of production. The wells must be judiciously distributed, which gives rise to a difficult problem of optimal control: how to choose the best installation sites of the production wells? This is precisely the question we deal within this work. These requirements lead us to the following objective functional:

$$J(u,p,f) = \frac{1}{2}\left\|u-U\right\|_2^2 + \frac{1}{2}\left\|p-P\right\|_2^2 + \frac{\beta_1}{2}\left\|f\right\|_{2q_0}^{2q_0}, \tag{2}$$

where $2 > q_0 > 1$ and $\beta_1 > 0$ is a coefficient of penalization. The first two terms in (2) make possible to minimize the difference between the reduced saturation of oil u, the global pressure p and the given data U and P, respectively. Our main goal is to present a method to carry out the optimal control of (1) with respect to all the important parameters arising in the process. More precisely, we seek necessary conditions for the admissible parameters u, p and f to minimize the functional J.

Theoretical analysis of the time-dependent dead oil problem with different types of boundary and initial conditions has received a significant amount of attention. See [4] for existence of weak solutions to systems related to (1), uniqueness and related regularity results in different settings with various assumptions on the data. So far, optimal control of a parabolic-elliptic dead oil system is studied in [15]. Optimal control of a discrete dead oil model is considered in [18]. Here we are interested to obtain necessary optimality conditions for the steady-state case. This is, to the best of our knowledge, an important open question.

Several techniques for deriving optimality conditions are available in the literature of optimal control systems governed by partial differential equations [8, 9, 10, 11, 13, 14]. In this work we apply the Lagrangian approach used with success by Bodart, Boureau and Touzani for an optimal control problem of the induction heating [2], and by Lee and Shilkin for the thermistor problem [7].

The motivation for our work is threefold. Firstly, the vast majority of the existing literature on dead oil systems deal with the parabolic-elliptic system. Considering that the relaxation time for the saturation of oil u is very small, the time derivative with respect to the saturation is dropped. Hence we get the system (1). Such a steady-state dead oil model represents a reasonably realistic situation where we neglect the time derivative. Secondly, some technical difficulties when dealing with system (1) arise and rely on the fact that there is no information on the time derivative of the reduced saturation of oil as well as on the pressure. As a result, one cannot use directly the standard compactness results to obtain strong convergence of sequences of solutions in appropriate spaces. This is in contrast with [15], where a fully parabolic system is considered. Thirdly, the choice of the cost function (2) for this time dependent problem seems to be quite appropriate from the point of view of practical applications.

The paper is organized as follows. In Section 2 we set up notation and hypotheses. Additionally, we recall two lemmas needed in the sequel. Our main results are stated and proved in the next two sections. Under adequate assumptions $(H1)$ and $(H2)$ on the data of the problem, existence and regularity of the optimal control are proved in Section 3. In Section 4, making use of the Lagrangian approach and assuming further the hypothesis $(H3)$, we derive necessary optimality conditions for a triple $\left(\bar{u}, \bar{p}, \bar{f}\right)$ to minimize (2) among all functions (u, p, f) verifying (1). We end with Section 5 of conclusion.

2 Preliminaries

The following assumptions are needed throughout the paper. Let g and d be real valued C^1-functions and φ be a C^3 function. It is required that

(H1) $0 < c_1 \leq d(r),\, g(r),\, \varphi(r) \leq c_2$; $c_3 \leq d'(r),\, \varphi'(r),\, \varphi''(r) \leq c_4$ for all $r \in \mathbb{R}$, where c_i, $i = 1, \ldots, 4$, are positive constants.

(H2) $U,\, P \in L^2(\Omega)$, where $U,\, P : \Omega \to \mathbb{R}$.

(H3) $|\varphi'''(r)| \leq c$ for all $r \in \mathbb{R}$.

Henceforth we use the standard notation for Sobolev spaces: we denote $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$ for each $p \in [1, \infty]$ and

$$W_p^1 = W_p^1(\Omega) := \{u \in L^p(\Omega),\, \nabla u \in L^p(\Omega)\}\,,$$

endowed with the norm $\|u\|_{W_p^1(\Omega)} = \|u\|_p + \|\nabla u\|_p$;

$$W_p^2 = W_p^2(\Omega) := \left\{u \in W_p^1(\Omega),\, \nabla^2 u \in L^p(\Omega)\right\}\,,$$

with the norm $\|u\|_{W_p^2(\Omega)} = \|u\|_{W_p^1(\Omega)} + \left\|\nabla^2 u\right\|_p$. Moreover, we set

$$
\begin{gathered}
V := W_2^1(\Omega); \\
W := \left\{ u \in W_{2q}^2(\Omega),\ u|_{\partial\Omega} = 0 \right\}, \\
\Upsilon := \left\{ f \in L^{2q}(\Omega) \right\}, \\
H := L^{2q}(\Omega) \times \overset{\circ}{W}{}_{2q}^{2-\frac{1}{q}}(\Omega),
\end{gathered}
$$

where $\overset{\circ}{W}{}_p^l(\Omega)$ is the interior of $W_p^l(\Omega)$.

In the sequel we use the following two lemmas in order to get regularity of solutions.

Lemma 2.1 ([12, 20])**.** *Let $\Omega \subset R^n$ be a bounded domain with a smooth boundary. Assume that $g \in (L^2(\Omega))^n$ and $a \in C(\bar{\Omega})$ with $\min_{\bar{\Omega}} a > 0$. Let u be the weak solution to the following problem:*

$$
\begin{aligned}
-\nabla \cdot (a \nabla u) &= \nabla \cdot g \quad \textit{in } \Omega \\
u &= 0 \quad \textit{on } \partial\Omega.
\end{aligned}
$$

Then, for each $p > 2$, there exists a positive constant c^*, depending only on n, Ω, a and p, such that if $g \in (L^2(\Omega))^n$, then

$$
\|\nabla u\|_p \le c^* \left(\|g\|_p + \|\nabla u\|_2 \right).
$$

Lemma 2.2 ([6])**.** *For any function $u \in C^\alpha(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega) \cap W_2^2(\Omega)$ there exist numbers N_0 and ϱ_0 such that for any $\varrho \le \varrho_0$ there is a finite covering of Ω by sets of the type $\Omega_\varrho(x_i)$, $x_i \in \bar{\Omega}$, such that the total number of intersections of different $\Omega_{2\varrho}(x_i) = \Omega \cap B_{2\varrho}(x_i)$ does not increase N_0. Hence, we have the estimate*

$$
\|\nabla u\|_4^4 \le c \|u\|_{C^\alpha(\Omega)}^2 \varrho^{2\alpha} \left(\left\|\nabla^2 u\right\|_2^2 + \frac{1}{\varrho^2} \|\nabla u\|_2^2 \right).
$$

3 Existence and Regularity of Optimal Solutions

In this section we prove existence and regularity of the optimal control under assumptions $(H1)$ and $(H2)$ on the data of the problem.

3.1 Existence of Optimal Solution

The following existence theorem is proved using Young's inequality together with the theorem of Lebesgue and some compactness arguments of Lions [8]. The existence follows from the fact that J is lower semicontinuous with respect to the weak convergence. Recall that along the text constants c are generic, and may change at each occurrence.

Theorem 3.1. *Under the hypotheses (H1) and (H2) there exists a* $q > 1$*, depending on the data of the problem, such that the problem of minimizing* (2) *subject to* (1) *has an optimal solution* $\left(\bar{u}, \bar{p}, \bar{f}\right)$ *satisfying*

$$\bar{u} \in W_q^2(\Omega) \cap L^2(\Omega),$$
$$\bar{p} \in L^2(\Omega) \cap W_{2q}^1(\Omega), \quad \bar{f} \in L^{2q_0}(\Omega).$$

Proof. Let $(u^m, p^m, f^m) \in W_2^1(\Omega) \times V \times L^{2q_0}(\Omega)$ be a sequence minimizing $J(u, p, f)$. Then we have that (f^m) is bounded in $L^{2q_0}(\Omega)$. By the second equation of (1) governed by the global pressure and a general result of elliptic PDEs [1], under our hypotheses we have that ∇p^m is bounded in $L^{2q}(\Omega)$. Writing now the first equation of (1) as

$$-\operatorname{div}\left(\varphi'(u^m)\varphi(u^m)\right) = \operatorname{div}\left(g(u^m)\nabla p^m\right)$$

and using Lemma 2.1, we obtain $\nabla u^m \in L^{2q}(\Omega)$. Hypotheses allow us to express again the first equation of (1) as

$$-\varphi'(u^m)\triangle u^m - \varphi''(u^m)|\nabla u^m|^2 = \operatorname{div}(g(u^m)\nabla p^m).$$

Hence,

$$\|u^m\|_{W_q^2(\Omega)} \leq c,$$

where all the constants c are independent of m. Using the Lebesgue theorem and compactness arguments of Lions [8], we can extract subsequences, still denoted by (p^m), (u^m) and (f^m), such that

$$u^m \to \overline{u} \text{ weakly in } W_q^2(\Omega),$$

$$p^m \to \overline{p} \text{ weakly in } W_{2q}^1(\Omega),$$

$$f^m \to \overline{f} \text{ weakly in } L^{2q_0}(\Omega).$$

Then, by Rellich's theorem, we have

$$p^m \to \overline{p} \text{ strongly in } L^2(\Omega).$$

Therefore, by using these facts and passing to the limit in problem (1), it follows from the weak lower semicontinuity of J with respect to the weak convergence, that the infimum is achieved at $\left(\overline{u}, \overline{p}, \overline{f}\right)$. □

3.2 Regularity of Solutions

Regularity of solutions given by Theorem 3.2 is obtained using Young's and Holder's inequalities, the Gronwall lemma, the De Giorgi-Nash-Ladyzhenskaya-Uraltseva theorem, an estimate from [5], and some technical lemmas that can be found in [6].

Theorem 3.2. *Let* $(\bar{u}, \bar{p}, \bar{f})$ *be an optimal solution to the problem of minimizing* (2) *subject to* (1)*. Suppose that (H1) and (H2) are satisfied. Then, there exist* $\alpha > 0$ *such that the following regularity conditions hold:*

$$\bar{u}, \bar{p} \in C^{\alpha}(\Omega), \tag{3}$$

$$\bar{u},\ \bar{p} \in W_4^1(\Omega), \tag{4}$$

$$\bar{u},\ \bar{p} \in W_2^2(\Omega), \tag{5}$$

$$\bar{u} \in C^{\frac{1}{4}}(\overline{\Omega}), \tag{6}$$

$$\bar{u} \in W_{2q_0}^2(\Omega), \quad \bar{p} \in W_{2q_0}^2(\Omega), \tag{7}$$

where q_0 *appears in the cost function* (2)*.*

Proof. Firstly, (3) is an immediate application of the general results of [6, 8, 19]. To continue the proof of Theorem 3.2, we need to estimate $\|\nabla u\|_4$ in function of $\|\nabla p\|_4$. Taking into account the first equation of (1), it is well known that $u \in W_4^1(\Omega)$ (see [5]) and

$$\|\nabla u\|_4 \leq c\|\nabla p\|_4. \tag{8}$$

Using Lemma 2.2, we have, for any $\varrho < \varrho_0$, that

$$\|\nabla p\|_4^4 \leq c\|p\|_{C^{\alpha}(\bar{\Omega})}^2 \varrho^{2\alpha} \left\{ \|\nabla p\|_4^4 + \frac{1}{\varrho^2} \|\nabla p\|_2^2 \right\}.$$

Therefore, we get (4) for an eligible choice of ϱ. Using (8), we obtain that $u \in W_4^1(\Omega)$. On the other hand, by the first equation of (1) and the regularity (4), we have that $u \in W_2^2(\Omega)$. Moreover, it follows, by the fact that $u \in W_2^2(\Omega)$, that $p \in W_2^2(\Omega)$. Using again (4) and the fact that $W_4^1(\Omega) \hookrightarrow C^{\frac{1}{4}}(\overline{\Omega})$, the regularity estimate (6) follows. Finally, the right-hand side of the first equation of (1) belongs to $L^4(\Omega) \hookrightarrow L^{2q_0}(\Omega)$ as $2q_0 < 4$. Thus, by (5) we get $u \in W_{2q_0}^2(\Omega)$. Since $f \in L^{2q_0}(\Omega)$, the same estimate follows from the second equation of the system (1) for p. □

4 Necessary Optimality Conditions

We define the following nonlinear operator corresponding to (1):

$$F : W \times W \times \Upsilon \longrightarrow H$$
$$(u, p, f) \longrightarrow F(u, p, f) = 0,$$

where

$$F(u, p, f) = \begin{pmatrix} -\Delta\varphi(u) - \operatorname{div}(g(u)\nabla p) \\ -\operatorname{div}(d(u)\nabla p) - f \end{pmatrix}.$$

Due to the estimate

$$\|v\|_{W^1_{\frac{4q}{2-q}}(\Omega)} \leq c\,\|v\|_{W^2_{2q}(\Omega)}, \quad \forall v \in W_{2q}^2(\Omega), \quad 1 < q < 2$$

(see [6]), hypothesis (H1) and regularity results (Theorem 3.2), we have

$$\varphi'(u)\Delta u, \varphi''(u)\left|\nabla u\right|^2, \quad g'(u)\nabla u \nabla p, \quad d(u)\nabla u \nabla p \in L^{\frac{2q}{2-q}}(\Omega) \subset L^{2q}(\Omega).$$

Thus, it follows that F is well defined.

4.1 Gâteaux Differentiability

Theorem 4.1. *Let assumptions (H1) through (H3) hold. Then, the operator F is Gâteaux differentiable and its derivative is given by*

$$\delta F(u,p,f)(e,w,h) = \frac{d}{ds}F\left(u+se, p+sw, f+sh\right)|_{s=0} = (\delta F_1, \delta F_2)$$
$$= \begin{pmatrix} -\operatorname{div}\left(\varphi'(u)\nabla e\right) - \operatorname{div}\left(\varphi''(u)e\nabla u\right) - \operatorname{div}\left(g(u)\nabla w\right) - \operatorname{div}\left(g'(u)e\nabla p\right) \\ -\operatorname{div}\left(d(u)\nabla w\right) - \operatorname{div}\left(d'(u)e\nabla p\right) - h \end{pmatrix}$$

for all $(e,w,h) \in W \times W \times \Upsilon$. Furthermore, for any optimal solution $\left(\bar{u},\bar{p},\bar{f}\right)$ of the problem of minimizing (2) among all the functions (u,p,f) satisfying (1), the image of $\delta F\left(\bar{u},\bar{p},\bar{f}\right)$ is equal to H.

To prove Theorem 4.1 we make use of the following lemma.

Lemma 4.2. *The operator $\delta F(u,p,f) : W \times W \times \Upsilon \longrightarrow H$ is linear and bounded.*

Proof. We have for all $(e,w,h) \in W \times W \times \Upsilon$ that

$$\delta_p F_2(u,p,f)(e,w,h) = -\operatorname{div}\left(d(u)\nabla w\right) - \operatorname{div}\left(d'(u)e\nabla p\right) - h$$
$$= -d(u)\triangle w - d'(u)\nabla u \cdot \nabla w - d'(u)e\triangle p - d'(u)\nabla e \cdot \nabla u - d'(u)e\nabla u \cdot \nabla p - h,$$

where $\delta_p F$ is the Gâteaux derivative of F with respect to p. Then, using hypothesis $(H1)$, we obtain that

$$\begin{aligned} &\|\delta_p F_2(u,p,f)(e,w,h)\|_{2q} \le \|\nabla w\|_{2q} + c\|\triangle w\|_{2q} \\ &\quad + c\|\nabla u \cdot \nabla w\|_{2q} + c\|e\triangle p\|_{2q} + c\|\nabla e \cdot \nabla u\|_{2q} + c\|e\nabla u \cdot \nabla p\|_{2q} + \|h\|_{2q}. \end{aligned} \tag{9}$$

We proceed to estimate the term $\|e\nabla u \cdot \nabla p\|_{2q}$. Similar arguments can be applied to the remaining terms of (9). We have

$$\begin{aligned} \|e\nabla u \cdot \nabla p\|_{2q} &\le \|e\|_\infty \|\nabla u \cdot \nabla p\|_{2q} \\ &\le \|e\|_\infty \|\nabla u\|_{\frac{4q}{2-q}} \|\nabla p\|_4 \\ &\le c\|u\|_W \|p\|_W \|e\|_W. \end{aligned}$$

Then,

$$\|\delta_p F_2(u,p,f)(e,w,h)\|_{2q} \le c\left(\|u\|_W, \|p\|_W, \|f\|_\Upsilon\right)\left(\|e\|_W + \|w\|_W + \|h\|_\Upsilon\right). \tag{10}$$

On the other hand,

$$
\begin{aligned}
&\delta_u F_1(u,p,f)(e,w,h)\\
&\quad = -\operatorname{div}\left(\varphi'(u)\nabla e\right) - \operatorname{div}\left(\varphi''(u)e\nabla u\right) - \operatorname{div}\left(g(u)\nabla w\right) - \operatorname{div}\left(g'(u)e\nabla p\right)\\
&\quad = -\varphi'(u)\triangle e - \varphi''(u)\nabla u\cdot\nabla e - \varphi''(u)e\triangle u - \varphi''(u)\nabla e\cdot\nabla u - \varphi'''(u)e|\nabla u|^2\\
&\qquad - g(u)\triangle w - g'(u)\nabla u\cdot\nabla w - g'(u)e\triangle p - g'(u)\nabla e\cdot\nabla p - g''(u)e\nabla u\cdot\nabla p,
\end{aligned}
$$

where $\delta_u F$ is the Gâteaux derivative of F with respect to u. The same arguments as above give that

$$
\|\delta_u F_1(u,p,f)(e,w,h)\|_{2q} \le c\left(\|u\|_W,\|p\|_W,\|f\|_\Upsilon\right)\left(\|e\|_W + \|w\|_W + \|h\|_\Upsilon\right). \tag{11}
$$

Hence, by (10) and (11),

$$
\|\delta F(u,p,f)(e,w,h)\|_{H\times H\times\Upsilon} \le c\left(\|u\|_W,\|p\|_W,\|f\|_\Upsilon\right)\left(\|e\|_W + \|w\|_W + \|h\|_\Upsilon\right).
$$

Consequently the operator $\delta_u F_1(u,p,f)$ is linear and bounded. □

Proof of Theorem 4.1. In order to show that the image of $\delta F(\overline{u},\overline{p},\overline{f})$ is equal to H, we need to prove that there exists a $(e,w,h)\in W\times W\times\Upsilon$ such that

$$
\begin{gathered}
-\operatorname{div}\left(\varphi'(\overline{u})\nabla e\right) - \operatorname{div}\left(\varphi''(\overline{u})e\nabla\overline{u}\right) - \operatorname{div}\left(g(\overline{u})\nabla w\right) - \operatorname{div}\left(g'(\overline{u})e\nabla\overline{p}\right) = \alpha,\\
-\operatorname{div}\left(d(\overline{u})\nabla w\right) - \operatorname{div}\left(d'(\overline{u})e\nabla\overline{p}\right) - h = \beta,\\
e|_{\partial\Omega} = 0,\\
w|_{\partial\Omega} = 0,
\end{gathered} \tag{12}
$$

for any $(\alpha,\beta)\in H$. Writing the system (12) for $h=0$ as

$$
\begin{gathered}
-\varphi'(\overline{u})\triangle e - 2\varphi''(\overline{u})\nabla\overline{u}\cdot\nabla e - \varphi''(\overline{u})e\triangle\overline{u} - \varphi'''(\overline{u})e|\nabla\overline{u}|^2\\
-g(\overline{u})\triangle w - g'(\overline{u})\nabla\overline{u}\cdot\nabla w - g'(\overline{u})e\triangle\overline{p} - g'(\overline{u})\nabla\overline{p}\cdot\nabla e - g''(\overline{u})e\nabla\overline{u}\cdot\nabla\overline{p} = \alpha,\\
-d(\overline{u})\triangle w - d'(\overline{u})\nabla\overline{u}\cdot\nabla w - d'(\overline{u})e\triangle\overline{p} - d'(\overline{u})\nabla\overline{u}\cdot\nabla\overline{e} - d'(\overline{u})e\nabla\overline{u}\cdot\nabla\overline{p} = \beta,\\
e|_{\partial\Omega} = 0,\\
w|_{\partial\Omega} = 0,
\end{gathered} \tag{13}
$$

it follows from the regularity of the optimal solution (Theorem 3.2) that

$$
\begin{gathered}
\varphi''(\overline{u})\triangle\overline{u},\ \varphi'''(\overline{u})|\nabla\overline{u}|^2,\ g'(\overline{u})\triangle\overline{p},\ g''(\overline{u})\nabla\overline{u}\cdot\nabla\overline{p},\ d'(\overline{u})\triangle\overline{p},\ d'(\overline{u})\nabla\overline{u}\cdot\nabla\overline{p}\in L^{2q_0}(\Omega),\\
\varphi''(\overline{u})\nabla\overline{u},\ g'(\overline{u})\nabla\overline{u},\ g'(\overline{u})\nabla\overline{p},\ d'(\overline{u})\nabla\overline{u}\in L^{4q_0}(\Omega).
\end{gathered}
$$

By general results of elliptic PDEs [6, 8, 19], there exists a unique solution of system (13) and hence there exists a $(e,w,0)$ verifying (12). We conclude that the image of δF is equal to H. □

4.2 Optimality Condition

We consider the cost functional $J : W \times W \times \Upsilon \to \mathbb{R}$ (2) and the Lagrangian $\mathcal{L}$ defined by

$$\mathcal{L}(u, p, f, p_1, e_1) = J(u, p, f) + \left\langle F(u, p, f), \begin{pmatrix} p_1 \\ e_1 \end{pmatrix} \right\rangle,$$

where the bracket $\langle \cdot, \cdot \rangle$ denote the duality between H and H'.

Theorem 4.3. *Under hypotheses (H1)–(H3), if* $(\overline{u}, \overline{p}, \overline{f})$ *is an optimal solution to the problem of minimizing* (2) *subject to* (1)*, then there exist functions* $(\overline{e_1}, \overline{p_1}) \in W_2^2(\Omega) \times W_2^2(\Omega)$ *satisfying the following conditions:*

$$\begin{gathered} \operatorname{div}(\varphi'(\overline{u})\nabla e_1) - d'(\overline{u})\nabla\overline{p} \cdot \nabla\overline{p_1} - \varphi''(\overline{u})\nabla\overline{u} \cdot \nabla\overline{e_1} - g'(\overline{u})\nabla\overline{p} \cdot \nabla\overline{e_1} = \overline{u} - U, \\ \overline{e_1}|_{\partial\Omega} = 0, \\ \operatorname{div}(d(\overline{u})\nabla\overline{p_1}) + \operatorname{div}(g(\overline{u})\nabla\overline{e_1}) = \overline{p} - P, \\ \overline{p_1}|_{\partial\Omega} = 0, \\ 2q_0\beta_1|\overline{f}|^{2q_0-2}\overline{f} = \overline{p_1}. \end{gathered} \tag{14}$$

Proof. Let $(\overline{u}, \overline{p}, \overline{f})$ be an optimal solution to the problem of minimizing (2) subject to (1). It is well known (cf., e.g., [3]) that there exist Lagrange multipliers $(\overline{p_1}, \overline{e_1}) \in H'$ verifying $\delta_{(u,p,f)}\mathcal{L}(\overline{u}, \overline{p}, \overline{f}, \overline{p_1}, \overline{e_1})(e, w, h) = 0$ for all $(e, w, h) \in W \times W \times \Upsilon$, with $\delta_{(u,p,f)}\mathcal{L}$ the Gâteaux derivative of $\mathcal{L}$ with respect to (u, p, f). We then obtain

$$\begin{aligned} &\int_\Omega \left((\overline{u} - U)e + (\overline{p} - P)w + 2q_0\beta_1|\overline{f}|^{2q_0-2}\overline{f}h\right) dx \\ &+ \int_\Omega \left(-\operatorname{div}(\varphi'(\overline{u})\nabla e) - \operatorname{div}(\varphi''(\overline{u})e\nabla\overline{u}) - \operatorname{div}(g(\overline{u})\nabla w) - \operatorname{div}(g'(\overline{u})e\nabla\overline{p})\right)\overline{e_1}\, dx \\ &\qquad + \int_\Omega \left(-\operatorname{div}(d(\overline{u})\nabla w) - \operatorname{div}(d'(\overline{u})e\nabla\overline{p}) - h\right)\overline{p_1}\, dx = 0 \end{aligned}$$

for all $(e, w, h) \in W \times W \times \Upsilon$. This last system is equivalent to

$$\begin{aligned} &\int_\Omega \Big((\overline{u} - U)e - \operatorname{div}(d'(\overline{u})e\nabla\overline{p})\,\overline{p_1} - \operatorname{div}(\varphi'(\overline{u})\nabla e)\,\overline{e_1} \\ &\qquad - \operatorname{div}(\varphi''(\overline{u})e\nabla\overline{u})\,\overline{e_1} - \operatorname{div}(g'(\overline{u})e\nabla\overline{p})\,\overline{e_1}\Big)\, dx \\ &+ \int_\Omega \left((\overline{p} - P)w - \operatorname{div}(d(\overline{u})\nabla w)\,\overline{p_1} - \operatorname{div}(g(\overline{u})\nabla w)\,\overline{e_1}\right) dx \\ &\qquad + \int_\Omega \left(2q_0\beta_1|\overline{f}|^{2q_0-2}\overline{f}h - \overline{p_1}h\right) dx = 0 \end{aligned}$$

for all $(e, w, h) \in W \times W \times \Upsilon$. In other words,

$$\int_\Omega \big((\overline{u} - U) + d'(\overline{u}) \nabla\overline{p} \cdot \nabla\overline{p_1} - \operatorname{div}(\varphi'(\overline{u})\nabla\overline{e_1}) + \varphi''(\overline{u}) \nabla\overline{u} \cdot \nabla\overline{e_1} + g'(\overline{u})\nabla\overline{p} \cdot \nabla\overline{e_1}\big) e\, dx + \int_\Omega ((\overline{p} - P) - \operatorname{div}(d(\overline{u})\nabla\overline{p_1}) - \operatorname{div}(g(\overline{u})\nabla\overline{e_1}))\, w\, dx + \int_\Omega \left(2q_0\beta_1|\overline{f}|^{2q_0-2}\overline{f}h - \overline{p_1}h\right)\, dx = 0 \quad (15)$$

for all $(e, w, h) \in W \times W \times \Upsilon$. Consider now the system

$$\begin{gathered} \operatorname{div}(\varphi'(\overline{u})\nabla e_1) - d'(\overline{u})\nabla\overline{p} \cdot \nabla p_1 - \varphi''(\overline{u})\nabla\overline{u} \cdot \nabla e_1 - g'(\overline{u})\nabla\overline{p} \cdot \nabla e_1 = \overline{u} - U, \\ \operatorname{div}(d(\overline{u})\nabla p_1) + \operatorname{div}(g(\overline{u})\nabla e_1) = \overline{p} - P, \\ e_1|_{\partial\Omega} = p_1|_{\partial\Omega} = 0. \end{gathered} \quad (16)$$

It follows again, by [6, 8, 19], that (16) has a unique solution $(e_1, p_1) \in W_2^2(\Omega) \times W_2^2(\Omega)$. Since the problem of finding $(e, w) \in W \times W$ satisfying

$$\begin{aligned} &-\operatorname{div}(\varphi'(\overline{u})\nabla e) - \operatorname{div}(\varphi''(\overline{u})e\nabla\overline{u}) - \operatorname{div}(g(\overline{u})\nabla w) - \operatorname{div}(g'(\overline{u})e\nabla\overline{p}) \\ &\quad = \operatorname{sign}(e_1 - \overline{e_1}) - \operatorname{div}(d(\overline{u})\nabla w) - \operatorname{div}(d'(\overline{u})e\nabla\overline{p}) \\ &\quad = \operatorname{sign}(p_1 - \overline{p_1}) \end{aligned} \quad (17)$$

is uniquely solvable on $W_{2q}^2 \times W_{2q}^2$, choosing $h = 0$ in (15), multiplying (16) by (e, w), integrating by parts, and making the difference with (15), we obtain

$$\begin{aligned} &\int_\Omega (-\operatorname{div}(\varphi'(\overline{u})\nabla e) - \operatorname{div}(\varphi''(\overline{u})e\nabla\overline{u}) - \operatorname{div}(g(\overline{u})\nabla w) - \operatorname{div}(g'(\overline{u})e\nabla\overline{p})) \\ &\times (e_1 - \overline{e_1})\, dx + \int_\Omega (-\operatorname{div}(d(\overline{u})\nabla w) - \operatorname{div}(d'(\overline{u})e\nabla\overline{p}))\, (p_1 - \overline{p_1})\, dx = 0 \end{aligned} \quad (18)$$

for all $(e, w) \in W \times W$. Choosing (e, w) in (18) as the solution of system (17), we have

$$\int_\Omega \operatorname{sign}(e_1 - \overline{e_1})(e_1 - \overline{e_1})\, dx + \int_\Omega \operatorname{sign}(p_1 - \overline{p_1})(p_1 - \overline{p_1})\, dx = 0.$$

It follows that $e_1 = \overline{e_1}$ and $p_1 = \overline{p_1}$. On the other hand, choosing $(e, w) = (0, 0)$ in (15), it follows (14), which concludes the proof of Theorem 4.3. □

5 Conclusion

In this paper, we considered the optimal control of a steady-state dead oil isotherm problem with Dirichlet boundary conditions, which is obtained from the standard parabolic-elliptic system, where the relaxation time for the reduced saturation of oil is very small. The main purpose was to prove existence and regularity of the optimal control and then necessary optimality conditions. The proposed method is based on the Lagrangian approach.

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